2018 SYDNEY BOYS HIGH SCHOOL
TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Extension 2

General Instructions

- Reading time – 5 minutes
- Working time – 3 hours
- Write using black pen
- Board-approved calculators may be used
- A reference sheet is provided with this paper
- Leave your answers in the simplest exact form, unless otherwise stated
- All necessary working should be shown in every question if full marks are to be awarded
- Marks may NOT be awarded for messy or badly arranged work
- In Questions 11–16, show relevant mathematical reasoning and/or calculations

Total marks – 100

Section I Pages 2–6

10 marks

- Attempt Questions 1–10
- Allow about 15 minutes for this section

Section II Pages 7–15

90 marks

- Attempt Questions 11–16
- Allow about 2 hours and 45 minutes for this section

Examiner: External

Section I

10 marks
Attempt Questions 1−10
Allow about 15 minutes for this section
Use the multiple-choice answer sheet for Questions 1–10.

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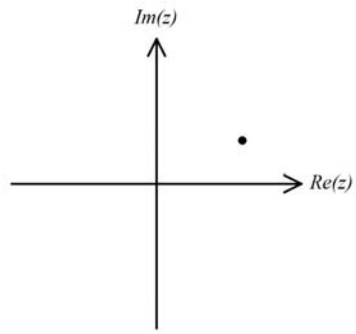
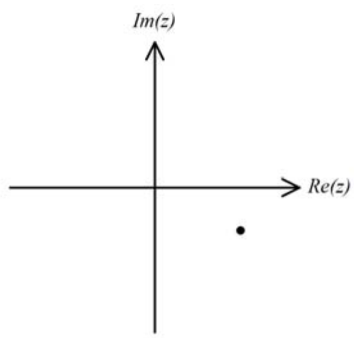
1 The complex number $x + iy$, where x and y are real constants, is represented in the following diagram.

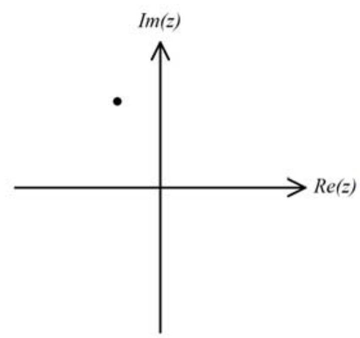


Which of the following (drawn to the same scale) could represent the complex number $ix - y$?

A.



B.



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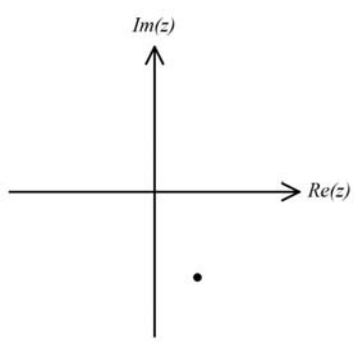
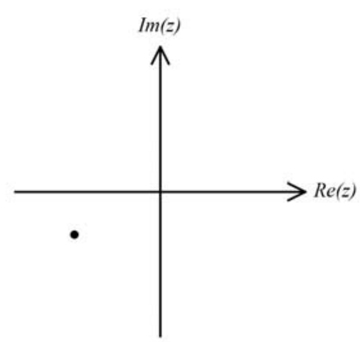
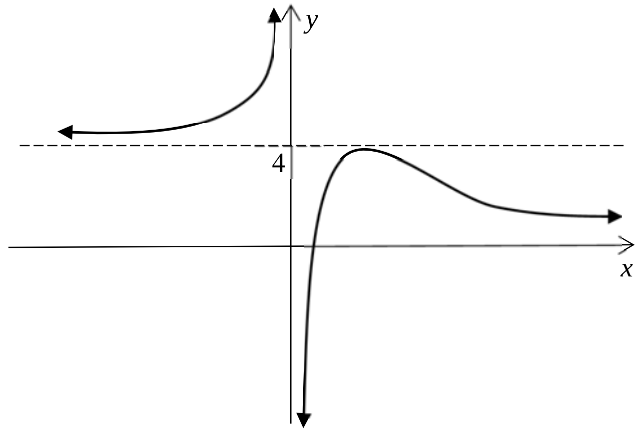


D.

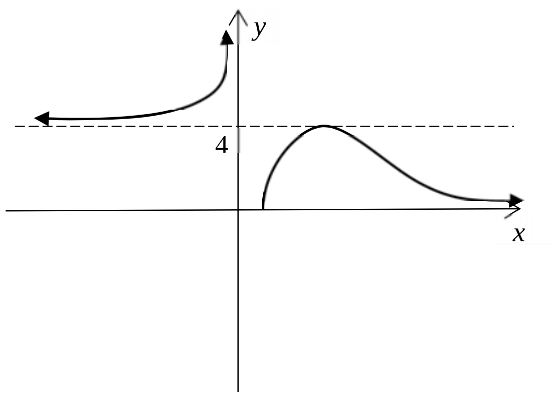


2 Below is the graph of $y = f(x)$.

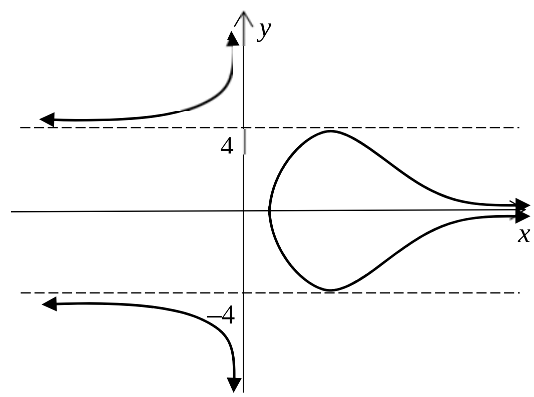


Which of the following could be the graph of $|y| = \sqrt{f(x)}$?

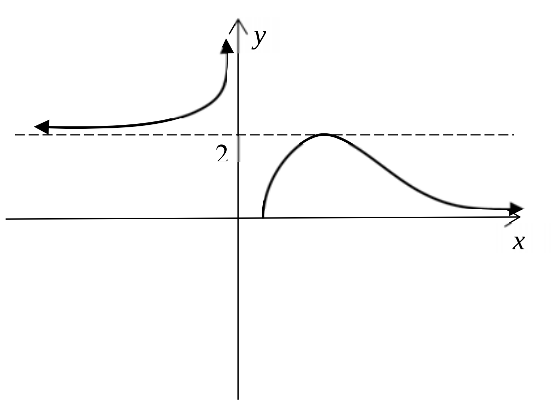
A.



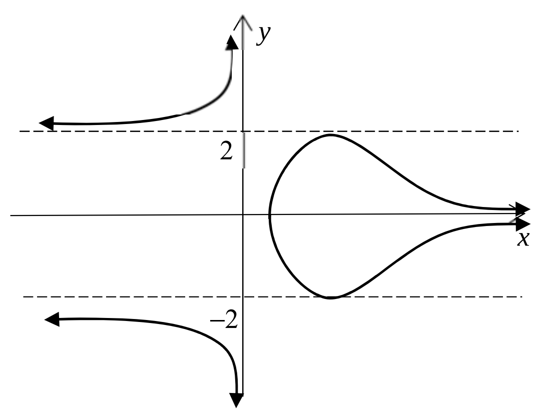
B.



C.



D.



3 The polynomial $P(x) = 2x^3 - 9x^2 + 12x + k$ has a double root.
What are the possible values of k?

A. $k = 4$ or 5

B. $k = -4$ or -5

C. $k = -4$ or 5

D. $k = 4$ or -5

4 If ω is a complex cube root of unity of least positive argument, what is the value of $\left(1+\frac{1}{\omega}\right)^{2018}$?

A. $\frac{1}{\omega}$

B. ω

C. 0

D. 1

5 Which of the following is equivalent to $\int_0^{\frac{\pi}{2}} \frac{dx}{3-\cos x+\sin x}$ using the substitution $t = \tan\frac{x}{2}$?

A. $\int_0^{\frac{\pi}{2}} \frac{dt}{2t^2+t+1}$

B. $\int_0^{\frac{\pi}{2}} \frac{2dt}{4-t^2-2t}$

C. $\int_0^{1} \frac{dt}{2t^2+t+1}$

D. $\int_0^{1} \frac{2dt}{4-t^2-2t}$

6 $P(x) = x^3 - ix^2 + 2x - 1$ has roots α, β and γ. What is the value of $\alpha^3 + \beta^3 + \gamma^3$?

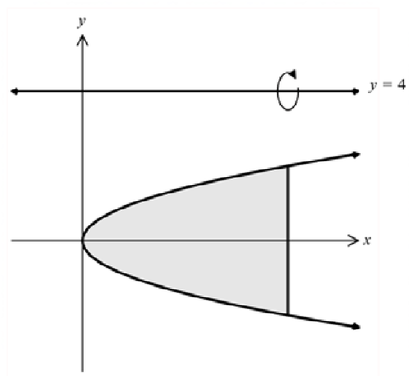
A. $3 + i$

B. $3 - 7i$

C. $1 + i$

D. $1 - 7i$

7 The shaded region bounded by the curve $x = y^2$ and the line $x = 4$ is rotated about the line $y = 4$.



Which of the following gives the expression for the volume of the solid of revolution using the method of cylindrical shells?

A. $2\pi \int_{-2}^{2} xy \, dy$

B. $2\pi \int_{-2}^{2} (4 - x)(4 - y) \, dy$

C. $4\pi \int_{0}^{2} xy \, dy$

D. $4\pi \int_{0}^{2} (4 - x)(4 - y) \, dy$

8 A horizontal force of P newtons causes a mass of m kg moving in a straight line to accelerate. The total resistance to the object's motion is kv^2 newtons per unit mass, where v is the speed of the object in m/s and k is a positive real constant. What is the equation of motion of the object?

A. $m\dfrac{dv}{dt} = P - kv^2$

B. $\dfrac{dv}{dt} = P - kv^2$

C. $m\dfrac{dv}{dt} = P - mkv^2$

D. $\dfrac{dv}{dt} = P - mkv^2$

9 Caleb has a jar of coins. There are ten coins each of 5 cent, 10 cent, 20 cent and 50 cent denominations. Caleb asks his friend Declan to choose eight coins from the jar.
In how many ways can Declan choose the eight coins?

A. 4^8

B. ${}^{40}C_8$

C. ${}^{11}C_3$

D. ${}^{7}C_3$

10 $f(x)$ is an even function. Which of the following is not necessarily true?

A. $\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(a-x)\,dx$

B. $\int_{0}^{2a} f(x)\,dx = \int_{-2a}^{0} f(-x)\,dx$

C. $\int_{0}^{2a} f(x)\,dx = \int_{0}^{a} f(a-x)\,dx + \int_{a}^{2a} f(2a-x)\,dx$

D. $\int_{a}^{2a} f(x)\,dx = \int_{0}^{a} f(a+x)\,dx$

Section II

Total marks − 90
Attempt Questions 11−16
Allow about 2 hour 45 minutes for this section.
Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

In Questions 11 to 16, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Use a SEPARATE writing booklet.

(a) Write $\dfrac{5}{2-4i}$ in the form $x+iy$. **1**

(b) Find the real and imaginary parts of $\left(i+\sqrt{3}\right)^5$. **2**

(c) Points A and B are representations in the complex plane of the numbers $z=1-i$ and $w=-\sqrt{3}-3i$ respectively. O is the origin.

(i) Find the size of angle AOB, expressing your answer in terms of π. **2**

(ii) Calculate the argument of zw, again giving your answer in terms of π. **1**

(d) Consider $f\left(x\right)=x^4+2x^3+2x^2+26x+169$.
The equation $f\left(x\right)=0$ has a root $x=2-3i$.

(i) Express $f\left(x\right)$ as a product of two real quadratic factors. **2**

(ii) Hence, or otherwise find all the roots of $f\left(x\right)=0$. **2**

(e) (i) On a single Argand diagram, sketch the following loci: **2**

(α) $\quad |z-3i|=2$

(β) $\quad \arg\left(z+1\right)=\dfrac{\pi}{4}$

(ii) On your diagram, shade the region where $|z-3i|\le 2$ and $\arg\left(z+1\right)\le\dfrac{\pi}{4}$. **1**

(iii) Indicate on your diagram, the point A on the locus $|z-3i|=2$ with the least argument and find this minimum argument. **2**

Question 12 (15 marks) Use a SEPARATE writing booklet.

(a) Find $\displaystyle\int \frac{dx}{\sqrt{2+2x-x^2}}$. **2**

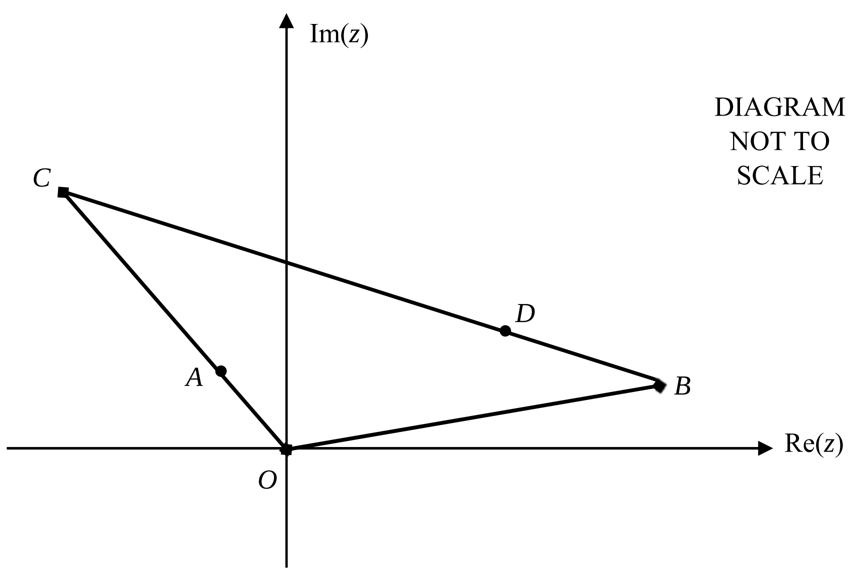
(b) Consider the curve $2x^3 + y^3 = 5y$.

(i) Show that $\dfrac{dy}{dx} = \dfrac{6x^2}{5-3y^2}$. **2**

(ii) Find the y-coordinates of the points where the curve has a horizontal tangent. **1**

(c) In the Argand diagram below, the point A represents the complex number z_1 and the point B represents the complex number z_2. **2**

C is chosen such that $\overrightarrow{OA} = \dfrac{1}{3}\overrightarrow{OC}$ and D is chosen such that $\overrightarrow{CD} = \dfrac{3}{4}\overrightarrow{CB}$.



Find $\overrightarrow{AD}$ in terms of z_1 and z_2 giving your answer in simplest form.
You must show all working.

Question 12 continues on Page 9

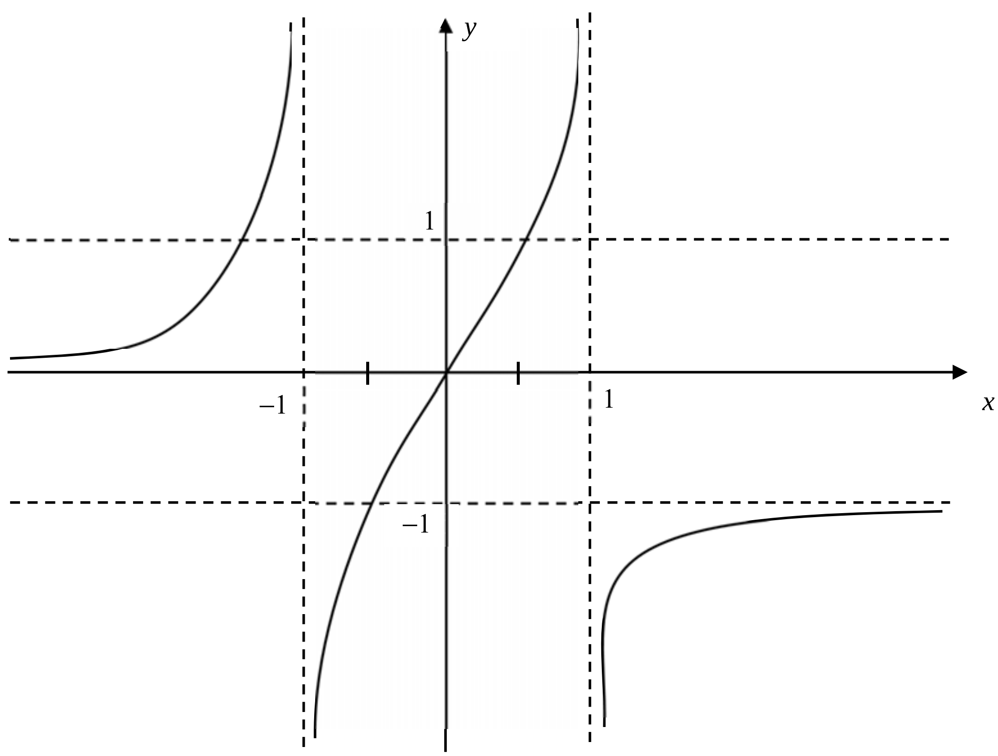
Question 12 (continued)

(d) Consider the curve $\mathscr{C}: y = \dfrac{x^2 + mx - n}{x+3}$.

(i) Show that if $\mathscr{C}$ has two stationary points, then $n < 9 - 3m$. **2**

(ii) Sketch the graph of $\mathscr{C}$ for $m = 2$ and $n = 1$, clearly showing where the asymptotes intersect the coordinate axes. **2**
You do not need to find the stationary points.

(e) The graph of $y = f(x)$ is shown below.



Draw a neat sketch of the following on the sheet provided

(i) $y = f(|x|)$ **2**

(ii) $y = \sin^{-1}(f(x))$ **2**

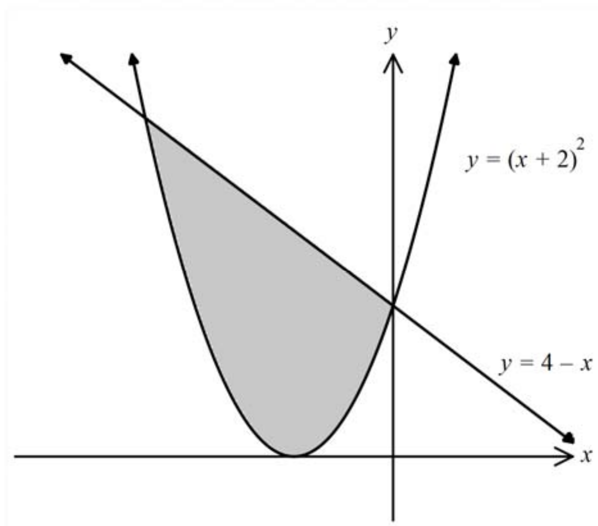
End of Question 12

Question 13 (15 marks) Use a SEPARATE writing booklet.

(a) The equation $x^3 + 2x^2 + x + 3 = 0$ has roots α, β and γ. Find the values of p, q and r such that $x^3 + px^2 + qx + r = 0$ has roots $\alpha\beta, \beta\gamma$ and $\gamma\alpha$. **3**

(b) Find $\int \tan^3 \theta \, d\theta$. **2**

(c) The diagram shows the region bounded by the curve $y = (x+2)^2$ and the line $y = 4 - x$. **4**



Find the volume generated when this region is rotated about the y-axis.

(d) Consider the integral $I_n = \int_0^1 \frac{x^n}{\sqrt{1-x}} dx$.

(i) Find I_0. **1**

(ii) Show that $I_{n-1} - I_n = \int_0^1 x^{n-1}\sqrt{1-x}\,dx$. **1**

(iii) Use integration by parts to show that $I_n = \frac{2n}{2n+1} I_{n-1}$. **2**

(iv) Hence, evaluate I_3. **2**

Question 14 (15 marks) Use a SEPARATE writing booklet.

(a) A toy car of mass 0.4 kg is set in motion with an initial velocity of 1 m/s. The resultant force acting on the car is $2-4v$ newtons, where v m/s is the velocity of the car t seconds after it is set in motion. Find how long it takes for the car's velocity to reduce to 0.55 m/s. **4**

(b) (i) Use De Moivre's Theorem to solve the equation $z^3 = -4+4\sqrt{3}i$. **2**

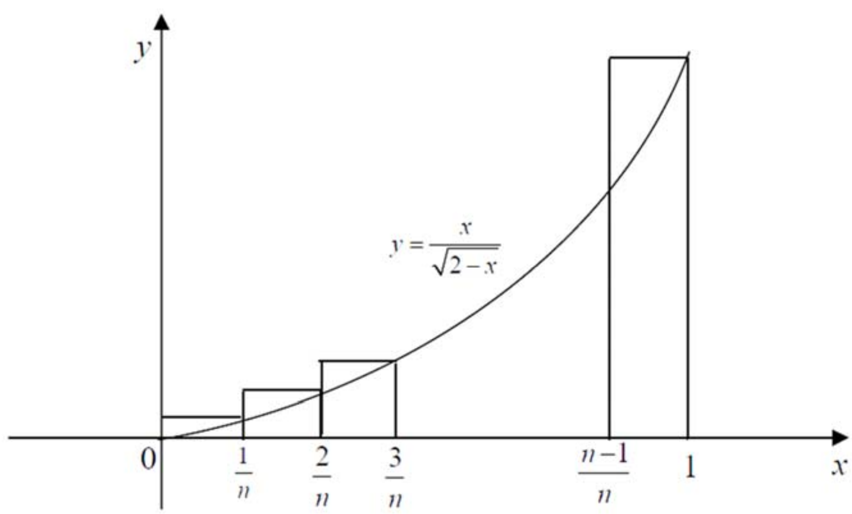
(ii) The roots of the equation $z^3 = -4+4\sqrt{3}i$ are represented by the points P, Q and R. Plot the roots on an Argand diagram and find the area of triangle PQR, giving your answer in exact form. **2**

(iii) By considering the roots of the equation $z^3 = -4+4\sqrt{3}i$, show that **2**

$$\cos\frac{2\pi}{9}+\cos\frac{4\pi}{9} = \cos\frac{\pi}{9}$$

(c) The diagram below shows the graph of $y=\dfrac{x}{\sqrt{2-x}}$ for $0 \le x \le 1$.

Rectangles of equal width are drawn as shown, in the interval between $x=0$ and $x=1$.



(i) Show that the total area of all the rectangles is given by **2**

$$S=\frac{1}{n\sqrt{n}}\left[\frac{1}{\sqrt{2n-1}}+\frac{2}{\sqrt{2n-2}}+\frac{3}{\sqrt{2n-3}}+....+\frac{n}{\sqrt{n}}\right]$$

(ii) As n increases, the width of the rectangles decreases. Find $\lim_{n\to\infty} S$, the limiting value of the total area of all rectangles. **3**

Question 15 (15 marks) Use a SEPARATE writing booklet.

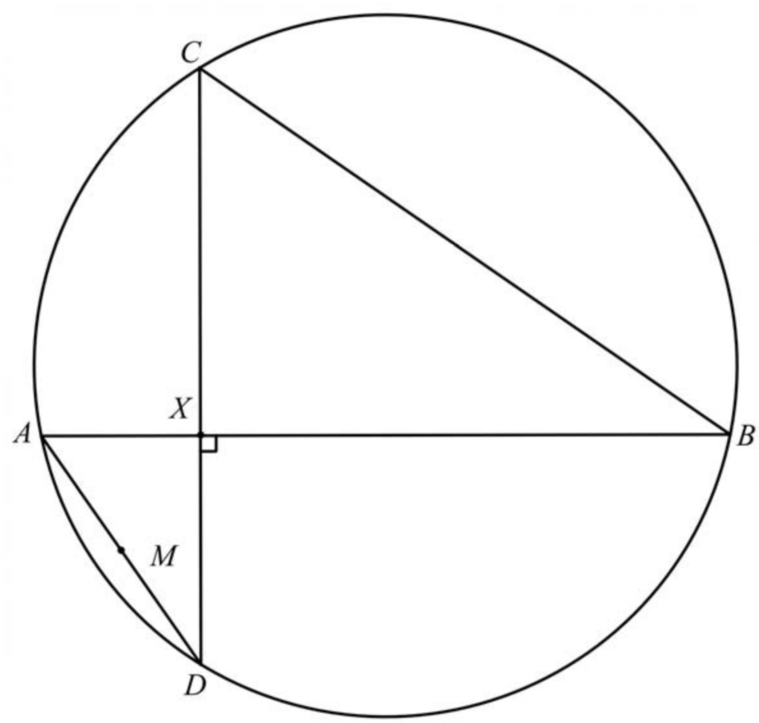
(a) Find $\int x^4 \ln x dx$. **2**

(b) (i) Prove that $a + b \geq 2\sqrt{ab}$ for $a, b \geq 0$. **1**

(ii) Hence, or otherwise, find the minimum value of the function **1**

$$f(x) = \frac{12x^2 \sin^2 x + 3}{x \sin x}$$ over the domain $0 < x < \pi$.

(c) AB and CD are perpendicular chords intersecting at X. **3**
M is the midpoint of AD. MX produced intersects BC at N.
Show that MN is perpendicular to BC.



Question 15 continues on Page 13

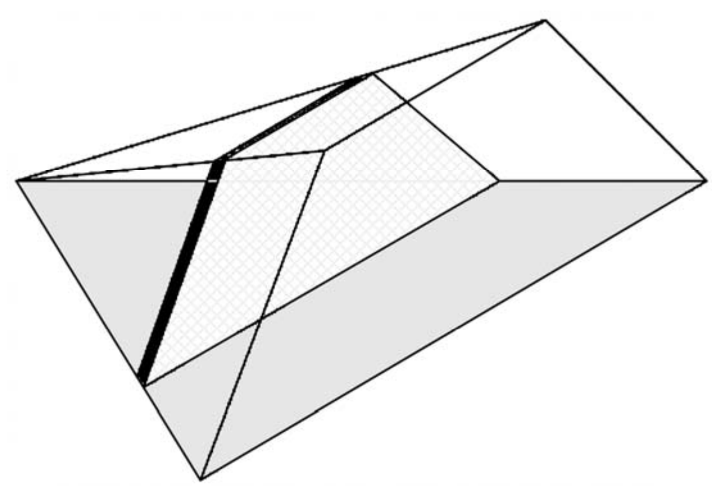
Question 15 (continued)

(d) (i) Explain why there are only two distinct ways to paint the faces of a cube such that three faces are red and three faces are blue. Rotations that result in the same colouring pattern are not considered distinct colourings. **1**

(ii) Find the number of ways to paint the cube, if each face is painted in one of two colours: red or blue. **3**

(e) The base of a solid is an equilateral triangle of side length 10 units. Cross-sections perpendicular to the base and parallel to one side of the triangle are trapeziums with the longer of the parallel sides in the base as shown. The lengths of the two parallel sides of the trapezium are in the ratio 2: 3 and the height of the trapezium is bounded by a plane inclined at 60° to the base. **4**

By considering the volume of a typical slice shown, use integration to find the volume of the solid.



End of Question 15

Question 16 (15 marks) Use a SEPARATE writing booklet.

(a) Consider $S_n(x) = e^{x^3} \dfrac{d^n}{dx^n}\left(e^{-x^3}\right)$, for integral $n \geq 1$.

(i) Find $S_1(x)$ and show that $S_2(x) = 9x^4 - 6x$, for integral $n \geq 1$. **1**

(ii) Use mathematical induction to prove that $S_n(x)$ is a polynomial in x. **2**

(iii) Write down the degree of this polynomial and the leading coefficient. **1**

(b) A skydiver jumps from an airplane and free-falls before opening his parachute.
The speed v of a skydiver t seconds after he opens the parachute can be modelled by the equation

$$\frac{dv}{dt} = -k\left(v^2 - p^2\right).$$

where p is a constant that depends on the type of the parachute, the mass of the skydiver and gravity.

(i) Show that the velocity of the skydiver is given by $v(t) = p\dfrac{\left(1 + Ae^{-2pkt}\right)}{\left(1 - Ae^{-2pkt}\right)}$, **3**

where A is a constant.

(ii) If the skydiver is falling at the rate of 10 m/s at the instant he opens the parachute, find the constant A in terms of p. **1**

(iii) For a particular skydiver, it is known that $p = 5$. **1**
Find the speed of the skydiver in terms of k and t.

(iv) Find the terminal velocity of this skydiver. **1**

Question 16 continues on Page 15

Question 16 (continued)

(c) Consider $f(x) = e^x(1+x^2)$.

(i) Show that $f'(x) \geq 0$ and, by sketching the graph of $f(x)$ or otherwise, explain why $e^x(1+x^2) = k$, where k is a constant, has exactly one real root if $k > 0$ and no real roots if $k \leq 0$. **2**

(ii) Hence or otherwise, find the number of real roots of the equation **3**

$$(e^x - 1) - k\tan^{-1} x = 0$$

when $0 < k \leq \frac{2}{\pi}$ and when $\frac{2}{\pi} < k < 1$ clearly justifying your answer.

End of paper

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Mathematics Extension 2

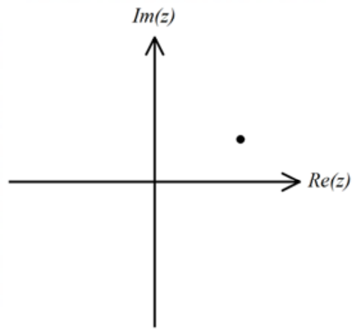
SUGGESTED SOLUTIONS

MC QUICK ANSWERS

1	B
2	D
3	B
4	A
5	C
6	B
7	B
8	C
9	C
10	C

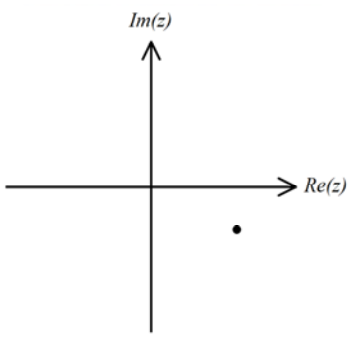
SECTION I MULTIPLE CHOICE SOLUTIONS

1 The complex number $x + iy$, where x and y are real constants, is represented in the following diagram.

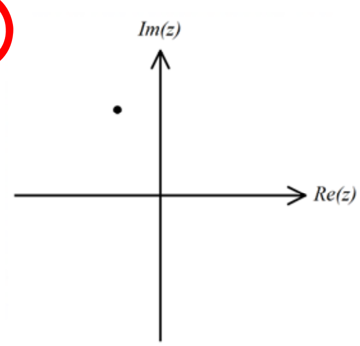


Which of the following (drawn to the same scale) could represent the complex number $ix - y$?

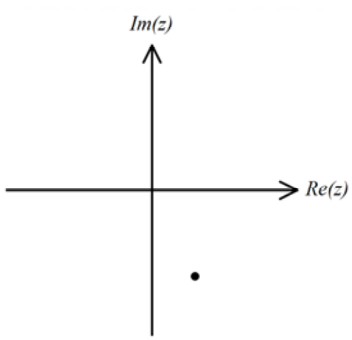
A.



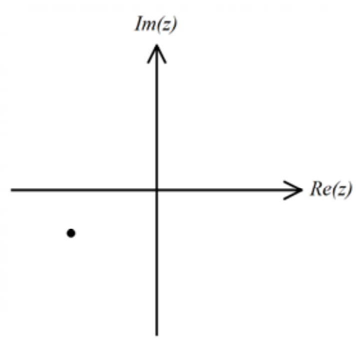
B.



C.



D.



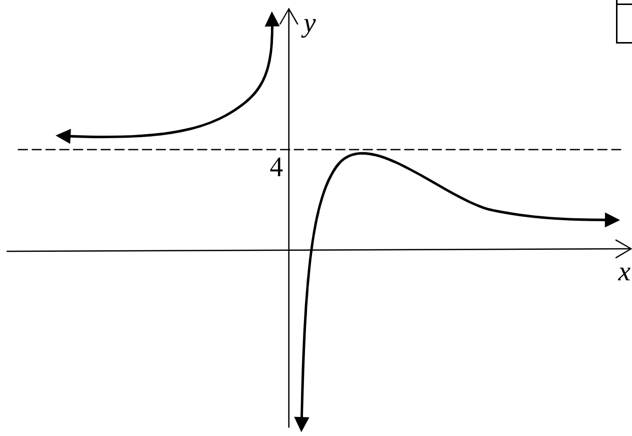
$ix - y = i(x + iy)$

iz is represented by rotating the point represented by z anticlockwise by 90°.

A	0
B	116
C	0
D	2

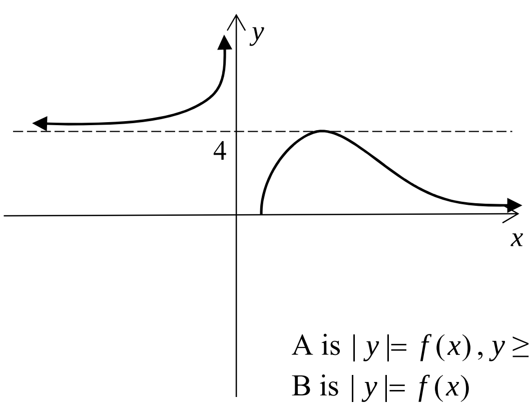
2 Below is the graph of $y = f(x)$.

A	0
B	2
C	5
D	111



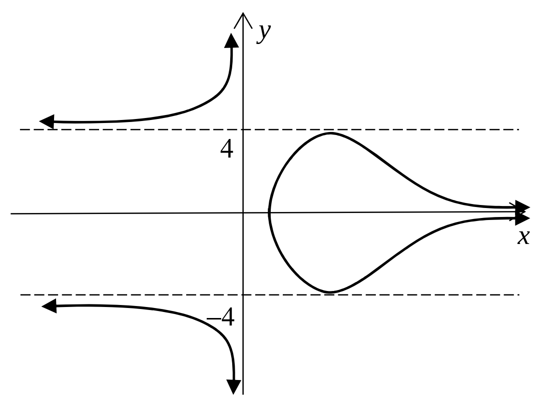
Which of the following could be the graph of $|y| = \sqrt{f(x)}$?

A.

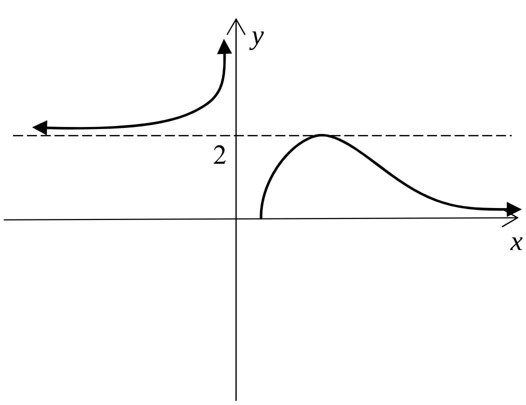


A is $|y| = f(x), y \geq 0$
B is $|y| = f(x)$
C is $y = \sqrt{f(x)}$

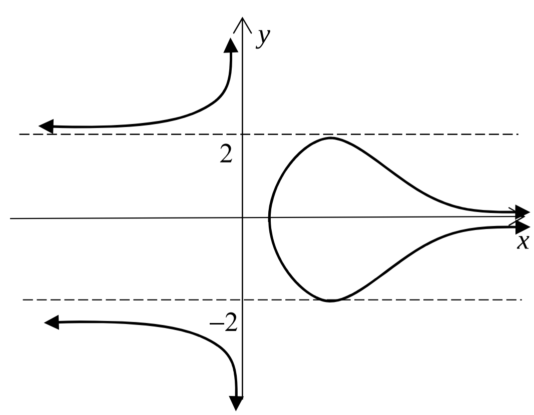
B.



C.



D.



3 The polynomial $P(x)=2x^3-9x^2+12x+k$ has a double root.
What are the possible values of *k*?

A. $k=4$ or 5

B. $k=-4$ or -5

C. $k=-4$ or 5

D. $k=4$ or -5

A	3
B	107
C	5
D	2

$$P'(x)=6x^2-18x+12$$
$$=6(x^2-3x+2)$$
$$=6(x-1)(x-2)$$

If $x=\alpha$ is a double root then $P(\alpha)=P'(\alpha)=0$
$P'(x)=0\Rightarrow x=1$ or 2
$P(1)=2-9+12+k=0\Rightarrow k=-5$
$P(2)=16-36+24+k=0\Rightarrow k=-4$

4 If ω is a complex cube root of unity of least positive argument, what is the value of $\left(1+\frac{1}{\omega}\right)^{2018}$?

A. $\frac{1}{\omega}$

B. ω

C. 0

D. 1

A	66
B	30
C	6
D	16

If ω is a complex cube root of unity then $1+\omega+\omega^2=0$ and $\omega^3=1$.
As well, $\omega^{-1}=\overline{\omega}=\omega^2$.

$$\left(1+\frac{1}{\omega}\right)^{2018}=\left(1+\omega^2\right)^{2018}=\left(-\omega\right)^{2018}=\omega^{3\times 224}\times\omega^2=\omega^2$$

5 Which of the following is equivalent to $\int_0^{\frac{\pi}{2}} \frac{dx}{3-\cos x+\sin x}$ using the substitution $t=\tan\frac{x}{2}$?

A. $\int_0^{\frac{\pi}{2}} \frac{dt}{2t^2+t+1}$

B. $\int_0^{\frac{\pi}{2}} \frac{2dt}{4-t^2-2t}$

C. $\int_0^{1} \frac{dt}{2t^2+t+1}$

D. $\int_0^{1} \frac{2dt}{4-t^2-2t}$

A	2
B	0
C	111
D	5

$t=\tan\frac{x}{2} \Rightarrow x=2\tan^{-1}t$

$\therefore dx=\frac{2\,dt}{1+t^2}$

$x: 0 \sim \frac{\pi}{2}$

$t: 0 \sim 1$

$$\int_0^{\frac{\pi}{2}} \frac{dx}{3-\cos x+\sin x}=\int_0^1 \frac{1}{3-\frac{1-t^2}{1+t^2}+\frac{2t}{1+t^2}}\times\frac{2\,dt}{1+t^2}$$

$$=\int_0^1 \frac{2\,dt}{3(1+t^2)-1+t^2+2t}$$

$$=\int_0^1 \frac{2\,dt}{4t^2+2t+2}$$

6 $P(x)=x^3-ix^2+2x-1$ has roots α,β and γ. What is the value of $\alpha^3+\beta^3+\gamma^3$?

A. $3+i$

B. $3-7i$

C. $1+i$

D. $1-7i$

A	15
B	85
C	5
D	13

$\sum\alpha=i;\ \sum\alpha\beta=2;\ \alpha\beta\gamma=1;\ \sum\alpha^2=\left(\sum\alpha\right)^2-2\sum\alpha\beta=(i)^2-2(2)=-5$

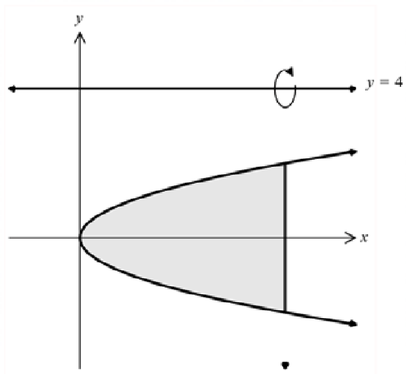
As $P(\alpha)=P(\beta)=P(\gamma)=0$

$$\left.\begin{array}{l}\alpha^3-i\alpha^2+2\alpha-1=0\\ \beta^3-i\beta^2+2\beta-1=0\\ \gamma^3-i\gamma^2+2\gamma-1=0\end{array}\right\}+$$

$\therefore \sum\alpha^3-i\sum\alpha^2+2\sum\alpha-3=0$

$\therefore \sum\alpha^3=i(-5)-2(i)+3=3-7i$

7 The shaded region bounded by the curve $x = y^2$ and the line $x = 4$ is rotated about the line $y = 4$.



$r = 4 - y$

$h = 4 - x$

$\delta V \doteqdot 2\pi r h \delta y$

$= 2\pi(4 - y)(4 - x)\delta y$

Which of the following gives the expression for the volume of the solid of revolution using the method of cylindrical shells?

A. $2\pi \int_{-2}^{2} xy \, dy$

B. $2\pi \int_{-2}^{2} (4-x)(4-y) \, dy$

C. $4\pi \int_{0}^{2} xy \, dy$

D. $4\pi \int_{0}^{2} (4-x)(4-y) \, dy$

A	2
B	87
C	2
D	27

8 A horizontal force of P newtons causes a mass of m kg moving in a straight line to accelerate. The total resistance to the object's motion is kv^2 newtons **per unit mass**, where v is the speed of the object in m/s and k is a positive real constant. What is the equation of motion of the object?

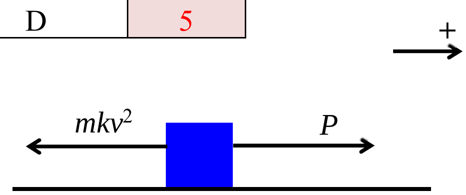
A. $m\frac{dv}{dt} = P - kv^2$

B. $\frac{dv}{dt} = P - kv^2$

C. $m\frac{dv}{dt} = P - mkv^2$

D. $\frac{dv}{dt} = P - mkv^2$

A	25
B	11
C	77
D	5



9 Caleb has a jar of coins. There are ten coins each of 5 cent, 10 cent, 20 cent and 50 cent denominations. Caleb asks his friend Declan to choose eight coins from the jar.
In how many ways can Declan choose the eight coins?

A. 4^8

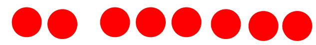
B. ${}^{40}C_8$

C. ${}^{11}C_3$

D. ${}^{7}C_3$

A	31
B	47
C	30
D	9

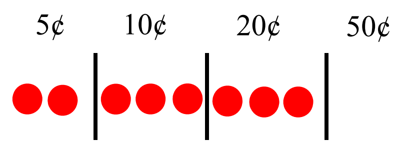
Start with all the coins being identical.
With the 8 coins we need 3 dividers to separate the coins.



Coins to the left of the 1st divider become 5¢ coins.
Then to the left of the next two dividers become 10¢ and 20¢ coins respectively.
To the right of the 3rd divider become the 50¢ coins.

With 8 identical coins and three identical dividers there are $\frac{11!}{8!\times 3!} = {}^{11}C_3$ ways.

For example:



This is the same as 2, 3, 3, 0

OR

Start with all the coins being identical.
With the 8 coins we need 3 dividers to separate the coins.

Coins to the left of the 1st divider become 5¢ coins.
Then to the left of the next two dividers become 10¢ and 20¢ coins respectively.
To the right of the 3rd divider become the 50¢ coins.

There are 9 positions for the first divider. Then there are 10 positions for the second divider and finally 11 positions for the third divider.
As the three dividers are identical, divide by 3!.

So there are $\frac{11\times 10\times 9}{3!} = {}^{11}C_3$ ways

10 $f(x)$ is an even function. Which of the following is not necessarily true?

A. $\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(a-x)\,dx$

B. $\int_{0}^{2a} f(x)\,dx = \int_{-2a}^{0} f(-x)\,dx$

C. $\int_{0}^{2a} f(x)\,dx = \int_{0}^{a} f(a-x)\,dx + \int_{a}^{2a} f(2a-x)\,dx$

D. $\int_{a}^{2a} f(x)\,dx = \int_{0}^{a} f(a+x)\,dx$

A	17
B	6
C	56
D	39

Note $\int_{0}^{a} f(x)\,dx = \int_{0}^{a} f(a-x)\,dx$

$$\begin{aligned}\int_{0}^{2a} f(x)\,dx &= \int_{0}^{a} f(x)\,dx + \int_{a}^{2a} f(x)\,dx \\ &= \int_{0}^{a} f(a-x)\,dx + \int_{a}^{2a} f(x)\,dx \\ &= \int_{0}^{a} f(a-x)\,dx + \int_{0}^{a} f(2a-x)\,dx\end{aligned}$$

X2 TRIAL

QUESTION 11

a $\frac{5}{2-4i} = \frac{5}{2-4i} \times \frac{2+4i}{2+4i}$

$= \frac{10(1+2i)}{20}$

$= \boxed{\frac{1}{2} + i}$

COMMENT: The question stated "Write in the form $x+iy$"

Very few failed to get the 1 mark.

b $(i+\sqrt{3})^5 = (\sqrt{3}+i)^5$

$= \left(2 \operatorname{cis} \frac{\pi}{6}\right)^5$

$= 32\left(\cos \frac{5\pi}{6} + i \sin \frac{5\pi}{6}\right)$

$= 32\left(-\frac{\sqrt{3}}{2} + \frac{1}{2}i\right)$

$= -16\sqrt{3} + 16i$

$\therefore$ Real Part $= -16\sqrt{3}$

Imaginary Part $= 16$

COMMENT Generally well done.

Common error was to interpret

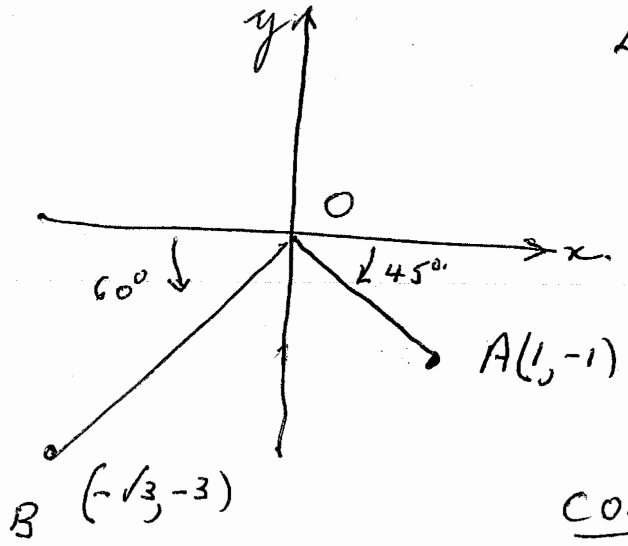
$i + \sqrt{3}$ as $1 + i\sqrt{3}$.

A few failed to simplify their answer despite clearly stated in the

GENERAL INSTRUCTIONS

• "Leave your answers in simplest exact form, unless otherwise stated."

(c) (i)



$\angle AOB = 180° - (45° + 60°)$

$= 75°$

$= \boxed{\frac{5\pi}{12}}$

COMMENT Well done.

(ii) $z = \sqrt{2} \text{ cis } \frac{-\pi}{4}$

$w = 2\sqrt{3} \text{ cis } \frac{-2\pi}{3}$

now $\arg(zw)$

$= \arg z + \arg w$

$= -\frac{\pi}{4} + \frac{-2\pi}{3}$

$= \boxed{-\frac{11\pi}{12}}$

COMMENT (c) (ii)

Accepted $\frac{13\pi}{12}$ (which is equivalent.)

for the 1 mark.

(d) (i) $x = 2 - 3i$ is a root of $f(x) = 0$.

$\therefore$ by conjugate root theorem.

$x = 2 + 3i$ is also a root (co-efficients are real)

$\therefore (x - (2-3i))(x - (2+3i))$ is a factor.

ie $x^2 - 4x + 13$ is a factor.

$$
\begin{array}{r}
x^2 + 6x + 13 \\
x^2 - 4x + 13 \overline{\smash{)}\, x^4 + 2x^3 + 2x^2 + 26x + 169} \\
\underline{x^4 - 4x^3 + 13x^2 \qquad\qquad\quad} \\
6x^3 - 11x^2 + 26x \qquad \\
\underline{6x^3 - 24x^2 + 78x \qquad} \\
13x^2 - 52x + 169 \\
\underline{13x^2 - 52x + 169} \\
0
\end{array}
$$

$\therefore$ $f(x) = (x^2 - 4x + 13)(x^2 + 6x + 13)$

COMMENT Well done. Of course there are other approaches to this question.

(ii) The roots of $x^2 + 6x + 13 = 0$.

$$x = \frac{-6 \pm \sqrt{36 - 52}}{2}$$

$$= \frac{-6 \pm 4i}{2}$$

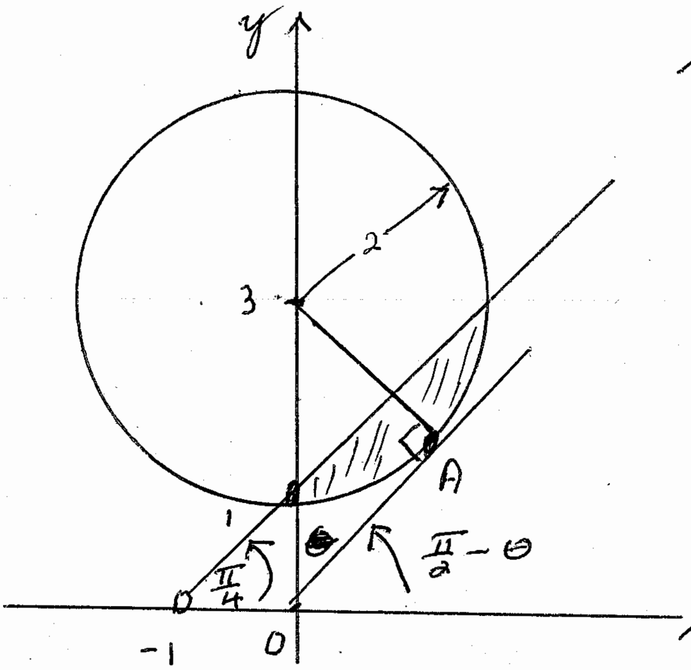
$$= -3 \pm 2i.$$

$\therefore$ The four roots are.

$$\boxed{2 \pm 3i, \ -3 \pm 2i}$$

COMMENT Well done.

(e).



now $\theta = \sin^{-1}\frac{2}{3}$

$\therefore$ least argument

is $\boxed{\frac{\pi}{2} - \sin^{-1}\frac{2}{3}}$

$\doteqdot \boxed{48°11'}$

COMMENT Common mistakes were.

(i) Wrong shading

(ii) A in the wrong place.

(iii) not using a single diagram.

Ext 2 Y12 THSC 2018 Q12 solutions

Mean (out of 15): 12.03

Q12(a) $\int \frac{dx}{\sqrt{2+2x-x^2}}$

$= \int \frac{dx}{\sqrt{3-(1-2x+x^2)}}$

$= \int \frac{dx}{\sqrt{3-(x-1)^2}}$

$= \sin^{-1}\frac{x-1}{\sqrt{3}} + C$ (2)

This was done quite well. A common error was considering $2+2x-x^2$ as $1-(x-1)^2$ rather than as $3-(x-1)^2$.

0	0.5	1	1.5	2	Mean
5	5	0	17	91	1.78

(b) (i) $2x^3 + y^3 = 5y$

$\therefore 6x^2 + 3y^2y' = 5y'$

$\therefore 6x^2 = y'(5-3y^2)$

$\therefore y' = \frac{6x^2}{5-3y^2}$ (2)

Very well done.

0	0.5	1	1.5	2	Mean
0	0	1	0	117	1.99

(ii) For horizontal tangent

$y' = 0$

$\therefore 6x^2 = 0$

$\therefore x = 0$

$\therefore 0 + y^3 = 5y$

$\therefore y(y^2-5) = 0$

$\therefore y = 0, \sqrt{5}, -\sqrt{5}$ (1)

Done well in general. Some looked at $5-3y^2=0$ and so were finding where the derivative was undefined rather than being equal to 0. Some, when solving $y^3 = 5y$, discarded $y = 0$.

0	0.5	1	Mean
15	15	88	0.81

(c) $\overrightarrow{AD} = \overrightarrow{AC} + \overrightarrow{CD}$

$= 2z_1 + \frac{3}{4}(z_2 - 3z_1)$

$= 2z_1 + \frac{3}{4}z_2 - \frac{9}{4}z_1$

$= \frac{3}{4}z_2 - \frac{1}{4}z_1$ (2)

There were a few approaches used to determine the required expression for $\overrightarrow{AD}$.

0	0.5	1	1.5	2	Mean
28	12	14	15	49	1.19

(d) $y = \frac{x^2+mx-n}{x+3}$

i) For stationary points $y'=0$

$y' = \frac{(x+3)(2x+m)-(x^2+mx-n)\times 1}{(x+3)^2}$

$= \frac{2x^2+mx+6x+3m-x^2-mx+n}{(x+3)^2}$

$= \frac{x^2+6x+3m+n}{(x+3)^2}$

If $y'=0$

$x^2+6x+3m+n = 0$

For 2 stationary points, $\Delta > 0$

$\therefore 6^2 - 4\times 1\times(3m+n) > 0$

$\therefore 36 - 12m - 4n > 0$

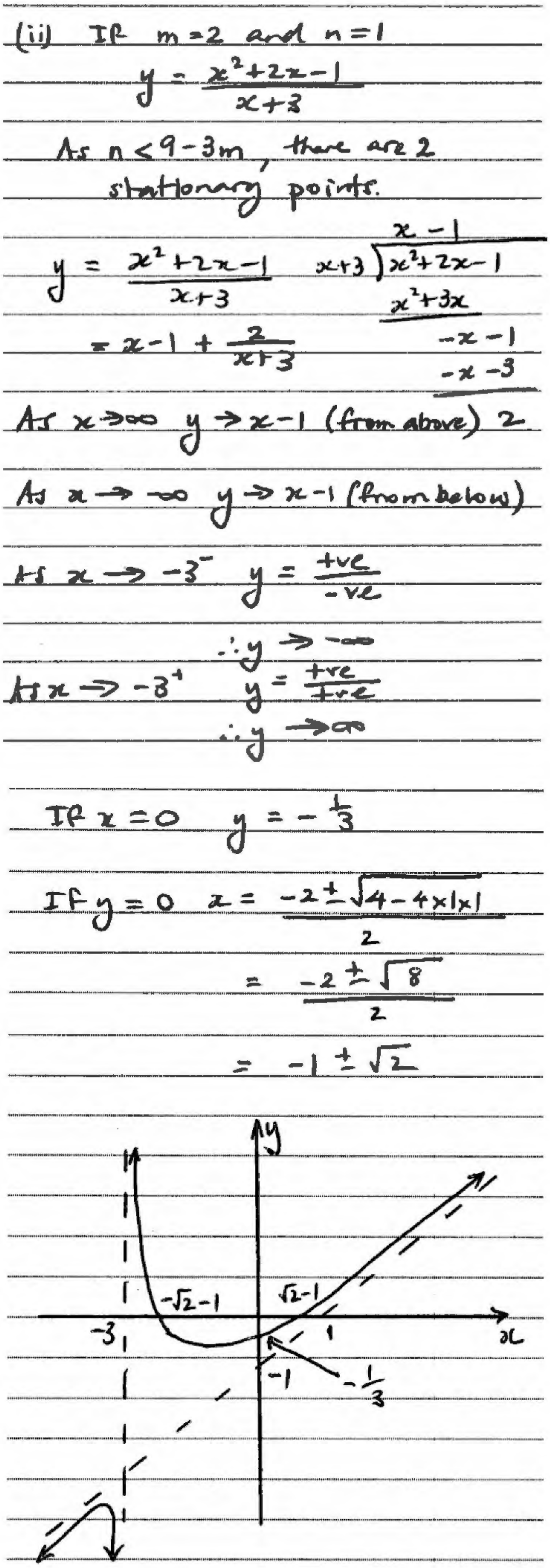
$\therefore 9 - 3m - n > 0$

$\therefore 9 - 3m > n$

$\therefore n < 9-3m$ (2)

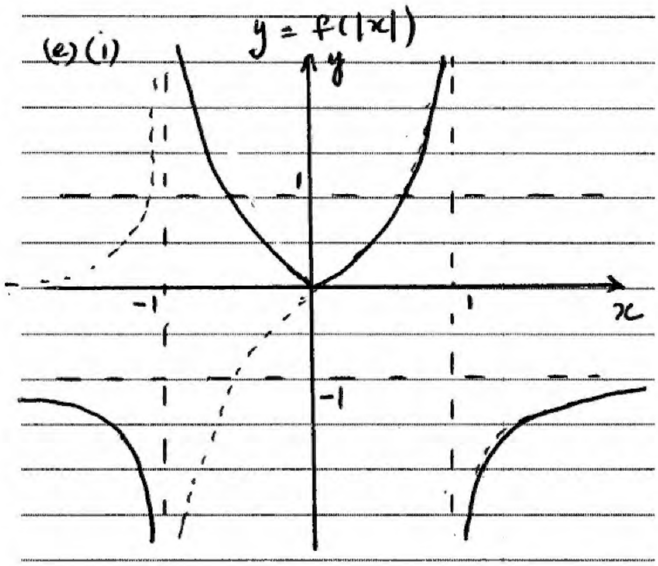
Most were able to determine the derivative and then consider the discriminant of the numerator to find the required condition for the existence of 2 stationary points.

0	0.5	1	1.5	2	Mean
4	11	13	1	89	1.68



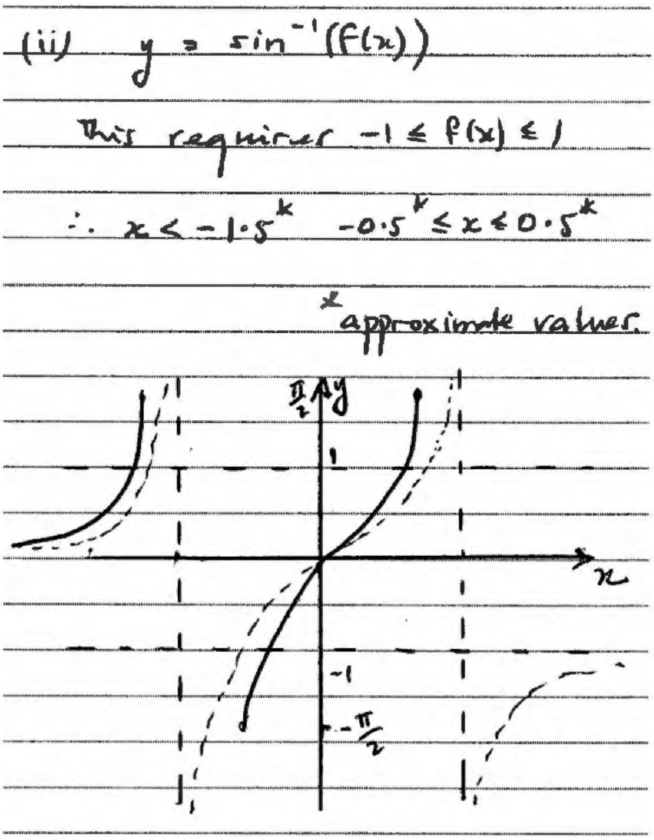
Some students did not see the relationship to part (i) which emphasises that 2 stationary points exist for the given values of *m* and *n*. Some did not try to find the *x* and *y* intercepts. Identifying $y = x - 1$ as an asymptote was reasonably well done, although some diagrams did not then sketch the graph as approaching this line at the extremities. Consideration of the curves behaviour on either side of $x = -3$ would have led to better sketches.

0	0.5	1	1.5	2	Mean
13	23	22	32	28	1.17



This was done quite well.

0	0.5	1	1.5	2	Mean
1	0	9	5	103	1.89



This was done quite well. A common error was not including the section of the sketch associated with the piece on the original diagram where $x < -1$. Other errors were not restricting sketches to the regions where $-1 \le f(x) \le 1$ and not restricting the range of the sketch to $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$.

0	0.5	1	1.5	2	Mean
8	5	16	33	56	1.53

13) a) Form equation with roots

$\alpha\beta$, $\beta\gamma$ and $\gamma\alpha$

ie $\frac{\alpha\beta\gamma}{\gamma}$, $\frac{\alpha\beta\gamma}{\alpha}$, $\frac{\alpha\beta\gamma}{\beta}$

since $\alpha\beta\gamma = -\frac{d}{a}$

$\alpha\beta\gamma = -3$

$-\frac{3}{\gamma}$, $-\frac{3}{\alpha}$, $-\frac{3}{\beta}$

let $X = -\frac{3}{x}$

$x = -\frac{3}{X}$

$$\left(-\frac{3}{X}\right)^3 + 2\left(-\frac{3}{X}\right)^2 + \left(-\frac{3}{X}\right) + 3 = 0$$

$$-\frac{27}{X^3} + \frac{18}{X^2} - \frac{3}{X} + 3 = 0$$

$$3X^3 - 3X^2 + 18X - 27 = 0$$

$$X^3 - X^2 + 6X - 9 = 0$$

$\therefore p = -1,\ q = 6,\ r = -9$

OR

$\alpha + \beta + \gamma = -\frac{b}{a}$

$\boxed{\alpha + \beta + \gamma = -2}$

$\alpha\beta + \alpha\gamma + \beta\gamma = \frac{c}{a}$

$\boxed{\alpha\beta + \alpha\gamma + \beta\gamma = 1}$

$\alpha\beta\gamma = -\frac{d}{a}$

$\boxed{\alpha\beta\gamma = -3}$

The roots of $x^3+px^2+qx+r=0$ are $\alpha\beta, \beta\gamma$ and $\alpha\gamma$.

$\alpha\beta+\beta\gamma+\alpha\gamma = -\frac{b}{a}$

$1 = -p$

$\therefore \underline{p=-1}$

$\alpha\beta.\beta\gamma + \alpha\beta.\alpha\gamma + \beta\gamma.\alpha\gamma = \frac{c}{a}$

$\alpha\beta\gamma(\alpha+\beta+\gamma) = q$

$-3(-2) = q$

$\therefore \underline{q = 6}$

$\alpha\beta.\beta\gamma.\alpha\gamma = -\frac{d}{a}$

$(\alpha\beta\gamma)^2 = -r$

$(-3)^2 = -r$

$\underline{r = -9}$

COMMENT:
Most students had an idea as to how to do the question. Students need to make sure they answer the question and take care with algebra.

b) $\int \tan^3\theta \, d\theta$

$= \int \tan^2\theta . \tan\theta \, d\theta$

$= \int (\sec^2\theta - 1)\tan\theta \, d\theta$

$= \int (\sec^2\theta.\tan\theta - \tan\theta)\, d\theta$

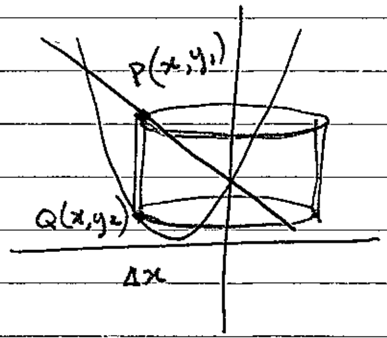
$= \int \left(\frac{1}{2}.2\sec^2\theta.\tan\theta + \frac{-\sin\theta}{\cos\theta}\right) d\theta$

$= \frac{1}{2}\tan^2\theta + \ln|\cos\theta| + C$

Note: other answers are valid such as $\frac{1}{2}\sec^2\theta - \ln|\sec\theta| + C$

COMMENT:
All students should be able to do this. Sadly not all could.

c)



$y = 4 - x$

$y = (x+2)^2$

$(x+2)^2 = 4 - x$

$x^2 + 4x + 4 = 4 - x$

$x^2 + 5x = 0$

$x(x+5) = 0$

$x = 0, -5$

$\Delta V = 2\pi r h \, \Delta x$

$\Delta V = 2\pi(-x)(y_1 - y_2)\Delta x$

$V = \lim_{\Delta x \to 0} \sum_{x=-5}^{0} 2\pi(-x)(4 - x - (x+2)^2)\Delta x$

$V = 2\pi \int_{-5}^{0} (-x)(4 - x - (x^2 + 4x + 4))\,dx$

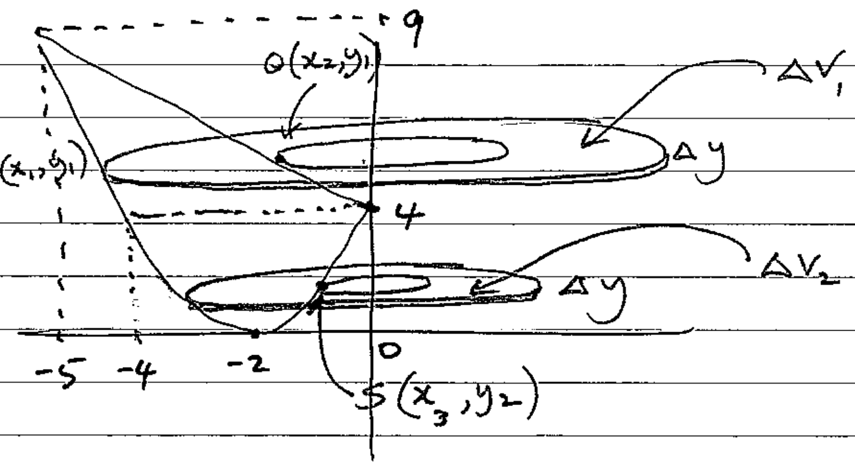
$V = 2\pi \int_{-5}^{0} (x^3 + 5x^2)\,dx$

$V = 2\pi \left[\frac{x^4}{4} + \frac{5x^3}{3} \right]_{-5}^{0}$

$V = 2\pi \left[(0) - \left(\frac{(-5)^4}{4} + \frac{5(-5)^3}{3} \right) \right]$

$V = \frac{625\pi}{6}$ cubic units.

OR



$y=(x+2)^2$

$x+2=\pm\sqrt{y}$

$x=-2\pm\sqrt{y}$

$x_1=-2-\sqrt{y}$

$x_3=-2+\sqrt{y}$

$$\Delta V_1 = \pi\left((-x_1)^2-(-x_2)^2\right)\Delta y$$

$$= \pi\left(x_1^2-x_2^2\right)\Delta y$$

$$V_1 = \lim_{\Delta y\to 0}\sum_{y=4}^{9}\pi\left((-2-\sqrt{y})^2-(4-y)^2\right)\Delta y$$

$$V_1 = \pi\int_4^9\left(4+4y^{\frac{1}{2}}+y-(16-8y+y^2)\right)dy$$

$$V_1 = \pi\int_4^9\left(-12+4y^{\frac{1}{2}}+9y-y^2\right)dy$$

$$V_1 = \pi\left[-12y+\frac{8}{3}y^{3/2}+\frac{9}{2}y^2-\frac{y^3}{3}\right]_4^9$$

$$V_1 = \pi\left[-12(9)+\frac{8}{3}(9)^{3/2}+\frac{9}{2}(9)^2-\frac{(9)^3}{3}-\left(-12(4)+\frac{8}{3}(4)^{3/2}+9\frac{(4)^2}{2}-\frac{(4)^3}{3}\right)\right]$$

$$V_1 = \frac{123\pi}{2}$$

$$\Delta V_2 = \pi\left((4+x_3)^2-(-x_3)^2\right)\Delta y$$

$$= \pi\left(16+8x_3+\cancel{x_3^2}-\cancel{x_3^2}\right)\Delta y$$

$$V_2 = \lim_{\Delta y\to 0}\sum_{y=0}^{4}8\pi\left(x_3+2\right)\Delta y$$

$$V_2 = 8\pi\int_0^4\left(-\cancel{2}+\sqrt{y}+\cancel{2}\right)dy$$

$$V_2 = 8\pi \int_0^4 y^{\frac{1}{2}} dy$$

$$V_2 = 8\pi \left[\frac{2}{3} y^{3/2} \right]_0^4$$

$$V_2 = 8\pi \left[\frac{2}{3}(4)^{3/2} - (0) \right]$$

$$V_2 = \frac{128\pi}{3}$$

$$V = V_1 + V_2$$

$$= \frac{123\pi}{2} + \frac{128\pi}{3}$$

$$= \frac{625\pi}{6}$$

COMMENT:

Students should know that cylindrical shells is the best method as ΔV is consistent across the range

Most students were unaware that the radius of the cylindrical shell is in fact $(-x)$.

As a result a lot of answers didn't match the working.

This question was done poorly.
A good diagram with clearly labelled points goes a long way!

d) i) $I_0 = \int_0^1 \frac{x^0}{\sqrt{1-x}}\, dx$

$= \int_0^1 (1-x)^{-\frac{1}{2}} dx$

$= \left.\frac{(1-x)^{\frac{1}{2}}}{(\frac{1}{2})(-1)}\right]_0^1$

$= \left[-2\sqrt{1-x}\right]_0^1$

$= -2\sqrt{1-(1)} - \left(-2\sqrt{1-(0)}\right)$

$= 2$

ii) $I_{n-1} - I_n = \int_0^1 \frac{x^{n-1}}{\sqrt{1-x}}\, dx - \int_0^1 \frac{x^n}{\sqrt{1-x}}\, dx$

$= \int_0^1 \frac{x^{n-1}(1-x)}{\sqrt{1-x}}\, dx$

$= \int_0^1 x^{n-1}\sqrt{1-x}\, dx$

iii) $I_{n-1} - I_n = \int_0^1 x^{n-1}\sqrt{1-x}\, dx$

$u = \sqrt{1-x}$ $\quad v' = x^{n-1}$

$u' = -\frac{1}{2\sqrt{1-x}}$ $\quad v = \frac{x^n}{n}$

$= \left[\frac{x^n}{n}\sqrt{1-x}\right]_0^1 + \frac{1}{2n}\int_0^1 \frac{x^n}{\sqrt{1-x}}\, dx$

$I_{n-1} - I_n = 0 + \frac{1}{2n} I_n$

$\left(\frac{1}{2n}+1\right) I_n = I_{n-1}$

$\left(\frac{2n+1}{2n}\right) I_n = I_{n-1}$

$$I_n = \frac{2n}{2n+1} I_{n-1}$$

OR

$$I_n = \int_0^1 \frac{x^n}{\sqrt{1-x}}\,dx$$

$u = x^n \qquad v' = \frac{1}{\sqrt{1-x}}$

$u' = nx^{n-1} \qquad v = -2\sqrt{1-x}$

$$I_n = \left[-2x^n\sqrt{1-x}\right]_0^1 + 2n\int_0^1 x^{n-1}\sqrt{1-x}\,dx$$

$$I_n = 0 + 2n\left[I_{n-1} - I_n\right] \qquad \text{from (ii)}$$

$$I_n = 2n I_{n-1} - 2n I_n$$

$$(2n+1) I_n = 2n I_{n-1}$$

$$I_n = \frac{2n}{2n+1} I_{n-1}$$

iv) $I_3 = \frac{2(3)}{2(3)+1} I_2$

$$= \frac{6}{7} I_2$$

$$= \frac{6}{7}\left(\frac{2(2)}{2(2)+1} I_1\right)$$

$$= \frac{6}{7}\left(\frac{4}{5} I_1\right)$$

$$= \frac{6}{7}\cdot\frac{4}{5}\cdot\frac{2(1)}{2(1)+1} I_0$$

$$= \frac{6}{7}\cdot\frac{4}{5}\cdot\frac{2}{3}\cdot(2) \qquad \text{from (i)}$$

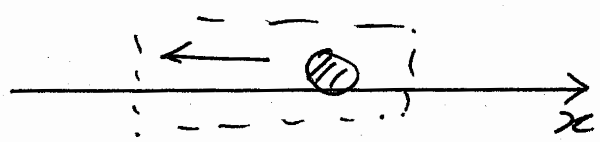
$$= \frac{32}{35}$$

COMMENT:

This question was done reasonably well. Even if they couldn't do all parts students picked up marks for the parts that they could do.

Solutions & Comments to Q14 ME2

(a)



$m = 0.4\ \text{kg}, \quad V = 1\ \text{m/s}.$

$\therefore \quad (0.4)\,\ddot{x} = 2 - 4V.$

$$\frac{dv}{dt} = \frac{2 - 4v}{0.4} = 5 - 10v.$$

[2] ie $\dfrac{dt}{dv} = \dfrac{1}{5(1-2v)}$

separating the variables.

$$\int dt = -\frac{1}{10}\int_0^{0.55}\left(\frac{-2}{1-2v}\right)dv$$

[2] $\therefore \quad t = \left[-\frac{1}{10}\ln|1-2v|\right]_0^{0.55}$

$$= \frac{1}{10}\ln 10 \doteqdot 0.23\ \text{sec}$$

(a) Comment:

Most students did well in this section as it should be for such problem.
Few students left it as $\frac{1}{10}\ln 10$ (No penalty) or $-\frac{1}{10}\ln\left(\frac{1}{10}\right)$.
Less than 10 students were not able to separate the variables and hence perform the correct integration.

Solution & Comments to Q14 ME 2 b(i) b(ii)

(b) (i)

$$z^3 = -4 + 4\sqrt{3}\,i$$
$$= 4(-1 + \sqrt{3}\,i)$$
$$= 8 \text{ cis}\left(\frac{2\pi}{3}\right).$$ [1]

$\therefore$ General solutions are:

$$z = 8^{\frac{1}{3}}\left[\text{cis}\left(\frac{2\pi}{3} + 2k\pi\right)\right]$$

i.e

$$z = 2\left[\cos\left(\frac{6k+2}{9}\right)\pi + i\sin\left(\frac{6k+2}{9}\right)\right]$$

for $k = -1, 0, 1.$ [1]

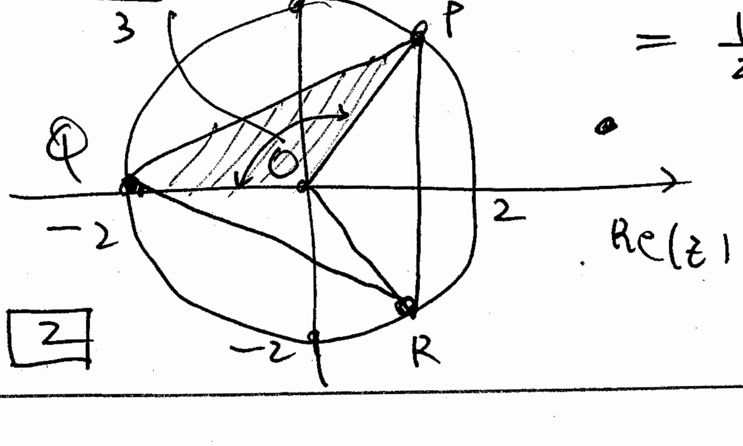
Comment

(b) (i) The roots are therefore

$2\text{ cis}\left(-\frac{4\pi}{9}\right)$, $2\text{ cis}\left(\frac{2\pi}{9}\right)$, $2\text{ cis}\left(\frac{8\pi}{9}\right)$

- Quite a few students did not know the roots are equally spaced on a circle of $r=2$ on the Argand diagram. ($\therefore$ penalty applied depends on working) but most did well.

(b) (ii)



- Area of $\triangle OPQ$
$= \frac{1}{2} \times 2^2 \times \sin\frac{2\pi}{3} = \sqrt{3}$
- $\therefore$ Area of $\triangle PQR$
$= 3 \times$ Area of $\triangle OPQ = 3\sqrt{3}$ (2)

(b) (ii) Comment:

Well done in this section

Few students try to find $|PQ|$ or $|PR|$ or $|QR|$ ..... but were not quite successful

(b) (iii) From (i)

$$z^3 - (-4+4\sqrt{3}i) = 0$$

$$z = 2\,\text{cis}\left(\frac{6k+2}{9}\right)\pi, \quad k=-1,0,1.$$

∴ roots are:

$$2\,\text{cis}\left(-\frac{4\pi}{9}\right),\ 2\,\text{cis}\left(\frac{2\pi}{9}\right)\ 2\,\text{cis}\left(\frac{8\pi}{9}\right)$$

Now $\sum \alpha_i = -\frac{b}{a} = 0$.

ie $2\left[\text{cis}\left(-\frac{4\pi}{9}\right) + \text{cis}\left(\frac{2\pi}{9}\right) + \text{cis}\left(\frac{8\pi}{9}\right)\right] = 0$

Equate real and imaginary pts:

[1] $\cos\left(-\frac{4\pi}{9}\right) + \cos\left(\frac{2\pi}{9}\right) + \cos\left(\frac{8\pi}{9}\right) = 0$

Using (1)

$\cos(-\theta) = \cos\theta$ ∴ $\cos\left(-\frac{4\pi}{9}\right) = \cos\left(\frac{4\pi}{9}\right)$

$\cos(\pi-\theta) = -\cos\theta$ $\cos\left(\frac{8\pi}{9}\right) = -\cos\left(\pi-\frac{8\pi}{9}\right)$

[1] and ↓ $= -\cos\frac{\pi}{9}$.

$$\therefore \cos\frac{2\pi}{9} + \cos\frac{4\pi}{9} - \cos\frac{\pi}{9} = 0$$

$$\therefore \cos\frac{2\pi}{9} + \cos\frac{4\pi}{9} = \cos\frac{\pi}{9}$$

(b) (iii) Comment:

Almost half of the students did not apply (1) and hence lose one mark. Some try to 'fudge' answers

Solutions & Comments to Q14 (c) (i), (ii)

(c) (i)

$$y = \frac{x}{\sqrt{2-x}}$$

Let S_n be the sum of the areas of the rectangles

S_1 = length (height) × width.

$$= \frac{\left(\frac{1}{n}\right)}{\sqrt{2-\left(\frac{1}{n}\right)}} \times \frac{1}{n} = \frac{1}{n\sqrt{n}}\left(\frac{1}{\sqrt{2n-1}}\right)$$

$$S_2 = \frac{\left(\frac{2}{n}\right)}{\sqrt{2-\left(\frac{2}{n}\right)}} \times \frac{1}{n} = \frac{1}{n\sqrt{n}}\left(\frac{2}{\sqrt{2n-2}}\right)$$

[1]

$$\vdots$$

$$S_n = \frac{1}{n\sqrt{n}} \times \frac{n}{\sqrt{2n-n}}$$ [1]

i.e.

$$S_n = \frac{1}{n\sqrt{n}}\left(\frac{1}{\sqrt{2n-1}} + \frac{2}{\sqrt{2n-2}} + \dots + \frac{n}{\sqrt{2n-n}}\right)$$

Comment:

Students has to show S_1 and S_2 ... S_n Inorder to get the first mark. Almost all of the students did well in this part

(c) (ii) The question is to find the limiting sum. In order to do this properly you need to show

- $\frac{1}{n}\sum_{r=0}^{n-1} f\left(\frac{r}{n}\right) < \int_0^1 f(x)\,dx < \frac{1}{n}\sum_{r=1}^{n} f\left(\frac{r}{n}\right)$
- or equivalent merit.

Solutions & comments to Q14 c (ii)

c (ii) • The interval [0, 1] is divided into n equal parts, each of width $h = \frac{1}{n}$.
Let the sum of inner rectangles be s, the outer rectangles be S, and the area between the curve $y=f(x)$ be A.

$$\therefore \quad s < A < S$$

$$s = \frac{1}{n}\left[f(0) + f\left(\frac{1}{n}\right) + \cdots + f\left(\frac{n-1}{n}\right)\right]$$

$$= \frac{1}{n}\left[f\left(\frac{0}{n}\right) + f\left(\frac{1}{n}\right) + \cdots + f\left(\frac{n-1}{n}\right)\right]$$

$$= \frac{1}{n}\sum_{r=0}^{n-1} f\left(\frac{r}{n}\right)$$

$$A = \int_0^1 f(x)\,dx$$

$$S = \frac{1}{n}\sum_{r=1}^{n} f\left(\frac{r}{n}\right)$$

$$\therefore \frac{1}{n}\sum_{r=0}^{n-1} f\left(\frac{r}{n}\right) < \int_0^1 f(x)\,dx < \frac{1}{n}\sum_{r=1}^{n} f\left(\frac{r}{n}\right)$$

(sandwich theorem).

• As $n \to \infty$ the area $\to \int_0^1 \frac{x}{\sqrt{2-x}}\,dx$ ~~from~~ above

[1] $$\therefore \lim_{n\to\infty} S_n = \int_0^1 \frac{x}{\sqrt{2-x}}\,dx$$

Let $u = 2-x$, $du = -dx$

When $x = 0$, $u = 2$

$x = 1$, $u = 1$

Solutions & Comments to Q (4 (c) (ii))

c (ii) $\lim_{n\to\infty} S_n = \int_2^1 \frac{2-u}{\sqrt{u}}\,du$

$= \int_1^2 \left(2u^{-1/2} - u^{\frac{1}{2}}\right) du$

$= \left[4u^{\frac{1}{2}} - \frac{2}{3}u^{3/2}\right]_1^2$

$= \sqrt{2}\left(\frac{12-4}{3}\right) - \frac{10}{3}$

$= \frac{8\sqrt{2}-10}{3}$ sq. units.

Comment on c(ii)

About 20% of the students did not realize as $n \to \infty$ the area $\to \int_0^1 \frac{x}{\sqrt{2-x}}\,dx$ from 'above'

- 1 mark is deduced if some comment were not made about the limiting sum as an integral.
- Students (some) has trouble in doing integration by substitution

Question 15 SOLUTIONS

(a)	Find $\int x^4 \ln x \, dx$.	2

$$\int x^4 \ln x \, dx = \int \frac{d}{dx}\left(\frac{1}{5}x^5\right) \ln x \, dx$$

$$= \frac{1}{5}x^5 \ln x - \int \frac{1}{5}x^5 \frac{d}{dx}\left(\ln x\right) dx$$

$$= \frac{1}{5}x^5 \ln x - \int \frac{1}{5}x^5 \times \frac{1}{x} \, dx$$

$$= \frac{1}{5}x^5 \ln x - \frac{1}{5}\int x^4 \, dx$$

$$= \frac{1}{5}x^5 \ln x - \frac{1}{5} \times \frac{1}{5}x^5 + C$$

$$= \frac{1}{5}x^5 \ln x - \frac{1}{25}x^5 + C$$

Comment

This was generally done well.

(b)	(i)	Prove that $a+b \geq 2\sqrt{ab}$ for $a, b \geq 0$.	1

Since $\left(\sqrt{a}-\sqrt{b}\right)^2 \geq 0$ then $a+b-2\sqrt{ab} \geq 0$

$\therefore a+b \geq 2\sqrt{ab}$

ALTERNATIVE

$$
\begin{aligned}
\text{LHS}-\text{RHS} &= a+b-2\sqrt{ab} \\
&= \left(\sqrt{a}+\sqrt{b}\right)^2 \\
&\geq 0 \\
\therefore \text{LHS} &\geq \text{RHS}
\end{aligned}
$$

Comment

This is the quickest way to do the problem without having any troublesome logic.

(ii)	Hence, or otherwise, find the minimum value of the function $f(x)=\dfrac{12x^2\sin^2 x+3}{x\sin x}$ over the domain $0<x<\pi$.	1

For $0<x<\pi$, $\sin x>0$ and so $x\sin x>0$

Minimum value = 12

Method 1:

$$
\begin{aligned}
\frac{12x^2\sin^2 x+3}{x\sin x} &\geq \frac{2\sqrt{12x^2\sin^2 x\times 3}}{x\sin x} \\
&= \frac{2\sqrt{36x^2\sin^2 x}}{x\sin x} \\
&= \frac{12x\sin x}{x\sin x} \\
&= 12
\end{aligned}
$$

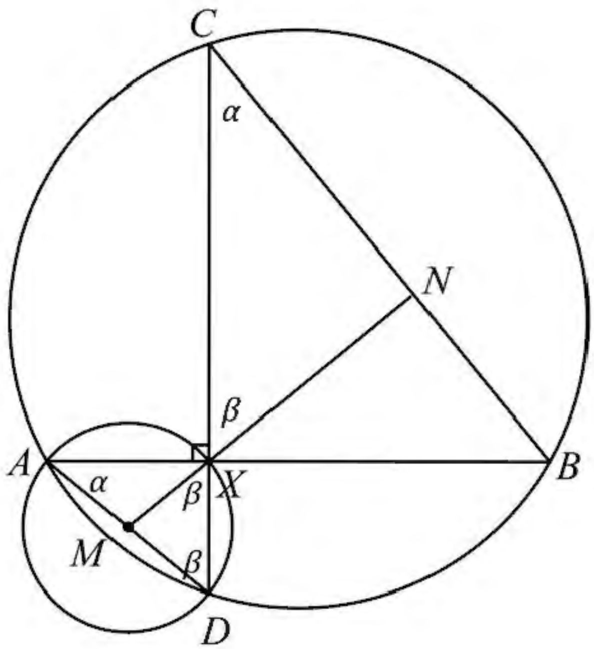
Method 2:

$$
\begin{aligned}
\frac{12x^2\sin^2 x+3}{x\sin x} &= 12x\sin x+\frac{3}{x\sin x} \\
&\geq 2\sqrt{12x\sin x\times\frac{3}{x\sin x}} \\
&= 2\sqrt{36} \\
&= 12
\end{aligned}
$$

Comment

Many students could not see the link from part (i).

(c)	*AB* and *CD* are perpendicular chords intersecting at *X*. *M* is the midpoint of *AD*. *MX* produced intersects *BC* at *N*. Show that *MN* is perpendicular to *BC*.	**3**



Construct *MXN*.

Construct circle *AXD* (**Note:** all triangles are concylic)
AD is a diameter of a circle through the vertices of ΔAXD (converse of angles in a semi-circle)

As *M* is the midpoint of *AD*, then *M* is the centre of circle *AXD*,
Let $\angle MAX = \alpha$ and $\angle MDX = \beta$.

$\therefore\ \alpha + \beta = 90°$ (angle sum of ΔAXD)

$MX = MD$ (radii of circle *AXD*)

$\therefore\ \angle MXD = \angle MDX = \beta$ (equal angles opposite equal sides)

$\therefore\ \angle CXN = \beta$ (vertically opposite angles)

$\angle XCN = \angle MAX = \alpha$ (angles in same segment, circle *ACB*)

$\therefore\ \angle XCN + \angle CXN = \alpha + \beta = 90°$

$\therefore\ \angle CNX = 90°$ (angle sum of ΔXCN)

$\therefore\ MN \perp CB$

Comment

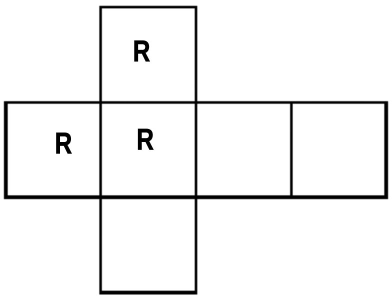
Too many students are wasting time verifying that all the angles at *X* are right angles. It was written that $AB \perp CD$.
Even allowing for legitimate abbreviations, students who wrote down abbreviations that did not make sense were penalised.
Students not referring to the "converse of the angle in a semi-circle" were penalised a ½ mark, but students who didn't even both to give a legitimate reason lost 1 mark.

Question 15 SOLUTIONS CONTINUED

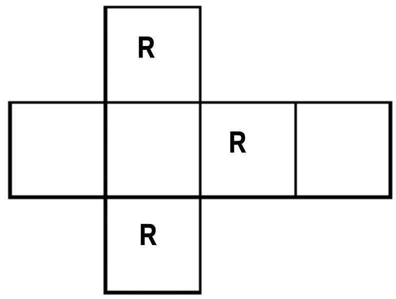
(d) (i) Explain why there are only two distinct ways to paint the faces of a cube such that three faces are red and three faces are blue. Rotations that result in the same colouring pattern are not considered distinct colourings. **1**

The two cases are when two red (or blue) sides are directly opposite or not.
Or consider when three faces meet at a common point or not.

(I)



(II)



Comment
Given that this was a "Show that …" question, there were many "manufactured" answers. These did not score well.

(ii) Find the number of ways to paint the cube, if each face is painted in one of two colours: red or blue. **3**

The following cases exist:

(i) 6R and 0B (6B and 0R)

There is only 1 way for 6R.
So there are 2 cases where there is only 1 colour.

(ii) 5R and 1B (5B and 1R)

Pick any face and paint it blue.
∴ there is only 1 case for 5R and 1B.
So there are 2 cases for either 1B or 1R.

(iii) 4R and 2B (4B and 2R)

Either the 2B are directly opposite or not.
So there are 4 cases for either 2B or 2R.

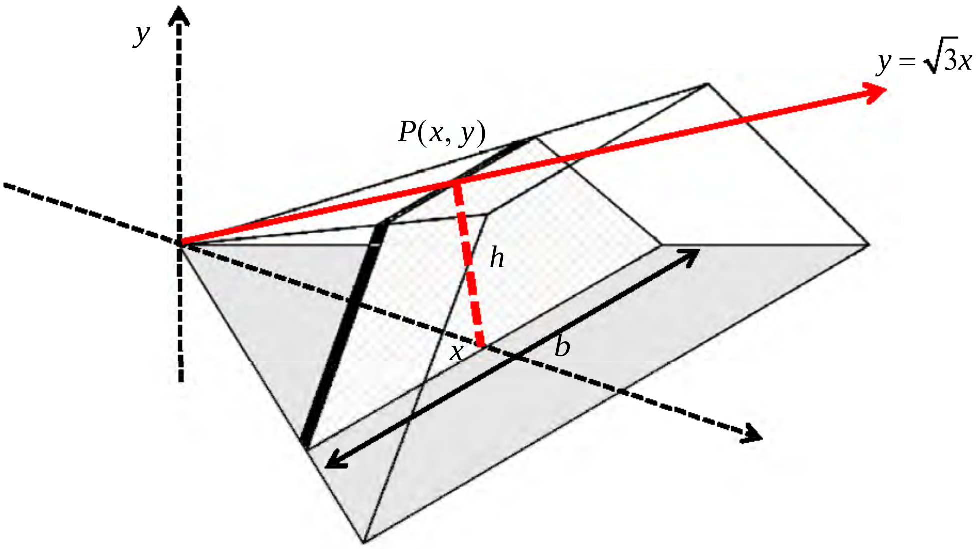
(iv) 3R and 3B

From (i), there are 2 cases.

∴ Total = 2 + 2 + 4 + 2 = 10

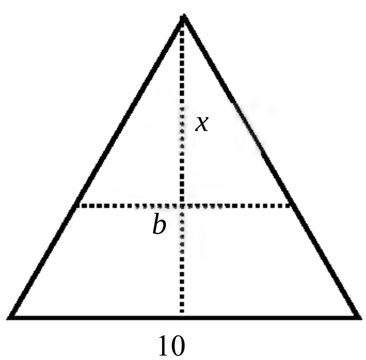
Comment
Generally the students who didn't manufacture an answer in part (i), were successful here.

(e) The base of a solid is an equilateral triangle of side length 10 units. Cross-sections perpendicular to the base and parallel to one side of the triangle are trapeziums with the longer of the parallel sides in the base as shown. The lengths of the two parallel sides of the trapezium are in the ratio 2: 3 and the height of the trapezium is bounded by a plane inclined at 60° to the base. By considering the volume of a typical slice shown, use integration to find the volume of the solid. **4**



At a distance x units the base of the trapezium is b and the height h, where $h=\sqrt{3}x$.
The upper length of the trapezium is $\frac{2}{3}b$.

The height of the base equilateral triangle is $5\sqrt{3}$ units (Pythagoras' Theorem).



x	0	$5\sqrt{3}$
b	0	10

By similarity, $b=\frac{10}{5\sqrt{3}}x=\frac{2}{\sqrt{3}}x$.

The volume of the slice

$$\Delta V \doteqdot \left(\frac{b+\frac{2}{3}b}{2}\right)h\,\Delta x$$

$$=\frac{5}{6}bh\,\Delta x$$

$$=\frac{5}{6}\times\frac{2}{\sqrt{3}}x\times\sqrt{3}x\,\Delta x$$

$$=\frac{5}{3}x^2\,\Delta x$$

(e) (continued)

The volume of the pyramid:

$$\begin{aligned} V &= \frac{5}{3}\int_0^{5\sqrt{3}} x^2\ dx \\ &= \frac{5}{3}\left[\frac{1}{3}x^3\right]_0^{5\sqrt{3}} \\ &= \frac{5}{9}\times\left(5\sqrt{3}\right)^3 \\ &= \frac{5}{9}\times 125\times 3\sqrt{3} \\ &= \frac{625\sqrt{3}}{3}\text{ cu} \end{aligned}$$

Comment

On the whole this was not done very well.
For those using the similarity approach, do NOT do a full similarity proof. Just give the relevant TLA.
The most common errors:

1. Getting the cross-section wrong in assuming the inclination of a side (non-parallel) of the trapezium to the base being 60°.

2. Using the same pronumeral for the sides of the trapezium as well as for the variable of integration and not making any adjustments i.e. calling the bottom $3x$ and the top $2x$ and then integrating for x: $0 \sim 5\sqrt{3}$ (or worse x: $0 \sim 10$)

3. Integrating along the side of the equilateral triangle (base) i.e. x: $0 \sim 10$

X2 TRIAL

QUESTION 16.

(a). Given $S_n(x) = e^{x^3} \frac{d^n}{dx^n}\left(e^{-x^3}\right)$; $n \geq 1$.

(i) $S_1(x) = e^{x^3} \cdot -3x^2 e^{-x^3}$

$= e^0 \cdot -3x^2$

$= -3x^2$

$S_2(x) = e^{x^3} \frac{d^2}{dx^2}\left(e^{-x^3}\right)$

$= e^{x^3} \cdot \frac{d}{dx} \cdot \frac{d}{dx}\left(e^{-x^3}\right)$

$= e^{x^3} \cdot \frac{d}{dx}\left[e^{-x^3} \times -3x^2\right]$

$= e^{x^3}\left[-6x e^{-x^3} + -3x^2 e^{-x^3} \times -3x^2\right]$

$= e^{x^3}\left[e^{-x^3}\left[-6x + 9x^4\right]\right]$

$= e^0\left(9x^4 - 6x\right)$

$= 9x^4 - 6x$

COMMENT: Well done. necessary for the rest of the question.

(II) Step I done in (I).

Step II Assume $S_k(x)$ is true

i.e. $e^{x^3} \cdot \frac{d^k}{dx^k}\left(e^{-x^3}\right) = P(x)$; where $P(x)$ is a Polynomial

Step III R.T.P.

$$S_{k+1}(x) = e^{x^3} \cdot \frac{d^{k+1}}{dx^{k+1}}\left(e^{-x^3}\right) = \varphi(x)$$

where $\varphi(x)$ is a Polynomial.

Now LHS $= S_{k+1}(x)$

$$= e^{x^3} \cdot \frac{d}{dx}\left[\frac{d^k}{dx^k} \cdot \left(e^{-x^3}\right)\right]$$

$$= e^{x^3} \cdot \frac{d}{dx}\left[\frac{P(x)}{e^{x^3}}\right] \quad \text{from the assumption}$$

$$= e^{x^3} \cdot \left[\frac{e^{x^3} \cdot P'(x) - P(x) \cdot 3x^2 e^{x^3}}{\left(e^{x^3}\right)^2}\right]$$

$$= \frac{\left(e^{x^3}\right)^2\left[P'(x) - 3x^2 P(x)\right]}{\left(e^{x^3}\right)^2}$$

$$= P'(x) - 3x^2 P(x) \quad \text{(clearly a Polynomial)}$$

$$= \varphi(x)$$

$$= RHS$$

STEP IV By the Principle of mathematical Induction, the statement is true for $n \geq 1$.

COMMENT. Proved difficult for many students.

The step involving the quotient rule was the issue for most.

(iii) deg $2n$ and leading co-efficient was $(-3)^n$.

COMMENT A few errors here.

(b) (i) $\frac{dv}{dt} = -k(v^2 - p^2)$

$$\frac{dt}{dv} = \frac{-1}{k(v-p)(v+p)}$$

$$= -\frac{1}{k}\left[\frac{A}{v-p} + \frac{B}{v+p}\right]$$

$$= -\frac{1}{k}\left[\frac{\frac{1}{2p}}{v-p} + \frac{-\frac{1}{2p}}{v+p}\right]$$ (This needs to be shown)

$$= \frac{-1}{2pk}\left[\frac{1}{v-p} - \frac{1}{v+p}\right]$$

now when $t = 0$ $v = v_0$ (say)

$$t = \frac{-1}{2pk} \ln \frac{v-p}{v+p} + c$$

$$0 = \frac{-1}{2pk} \ln \frac{v_0 - p}{v_0 + p} + c$$

$$C = \frac{1}{2pk} \ln D$$ ($\ln \frac{v_0 - p}{v_0 + p}$ is a constant)

$$\therefore \quad t = \frac{-1}{2pk} \ln \frac{v-p}{v+p} + \frac{1}{2pk} D$$

$$-2pkt = \ln\left(\frac{v-p}{v+p}\right) + D.$$

$$\ln \frac{v-p}{v+p} = e^{-2pkt} \times e^{-D}.$$

$$\ln\left(\frac{v-p}{v+p}\right) = A\, e^{-2pkt} \quad (\text{where } e^{-D} = A)$$

$$\therefore \frac{v-p}{v+p} = A\, e^{-2pkt}$$

$$v - p = (v+p) A e^{-2pkt}$$

$$v(1 - Ae^{-2pkt}) = p(Ae^{-2pkt} + 1)$$

$$\boxed{v = p\left(\frac{1 + Ae^{-2pkt}}{1 - Ae^{-2pkt}}\right)}$$

(as required)

<u>COMMENT</u> many were confused by 'A' and merely inserted it along the way without proper explanation.

(ii) $$10 = p\left(\frac{1+A}{1-A}\right) \quad (\text{when } t = 0 \quad e^{-2pk.0} = 1)$$

$$10(1-A) = p(1+A)$$

$$10 - 10A = p + pA$$

$$A(p+10) = 10 - p$$

$$\boxed{A = \frac{10-p}{10+p}}$$

<u>COMMENT</u>

Well done.

(iii) $A = \dfrac{10-5}{10+5}$

$= \boxed{\dfrac{1}{3}}$

COMMENT. Easy and well done.

(~~iv~~) $\therefore \boxed{v = 5\dfrac{\left(1+\frac{1}{3}e^{-10kt}\right)}{\left(1-\frac{1}{3}e^{-10kt}\right)}}$

COMMENT Well done.

(iv)

as $t \to \infty$.

$v \to 5\dfrac{\left(1+\frac{1}{3}\times 0\right)}{\left(1-\frac{1}{3}\times 0\right)}$

$= \boxed{5 \text{ m/s}}$

COMMENT. Could be done using

$\dfrac{dv}{dt} \to 0$. ; $v \to P$
$= 5$ m/s.

(c) given $f(x) = e^x(1+x^2)$

(i) $f'(x) = e^x(2x) + e^x(1+x^2)$

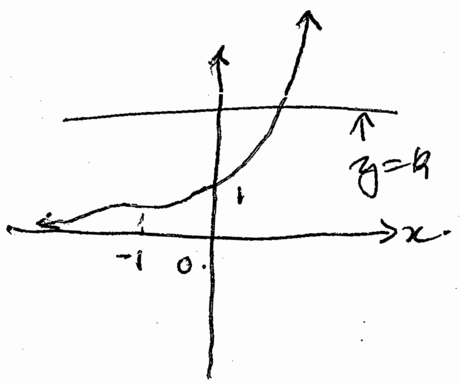
$= e^x(1+2x+x^2)$

$= e^x(1+x)^2$

$\geq 0.$

Because $e^x(1+x^2)$ is increasing

$e^x(1+x^2) = k.$



as one solution if $k>0$ and none if $k \leq 0$. Since $f(x) > 0$ for all x.

COMMENT most were able to satisfactorily handle this question.

(ii). given $(e^x - 1) - k\tan^{-1}x = 0.$

Let $F(x) = (e^x - 1) - k\tan^{-1}x = 0$ (A)

now $F'(x) = e^x - \frac{k}{1+x^2} = 0$

ie $e^x = \frac{k}{1+x^2}.$

$e^x(1+x^2) = k.$

The implication from (1) is

that $F'(x) = 0$ has one root for $k > 0$.

ie. $F(x)$ has one st. point

Also $F(0) = 0$ $\therefore$ $F(x)$ has a root at $x = 0$

and from (A) $e^x - 1 = k \tan^{-1} x$, (B)

We know that

$$-\frac{\pi}{2} < \tan^{-1} x < \frac{\pi}{2}$$ (C)

Consider $0 < k < \frac{2}{\pi}$.

ie $0 < k \tan^{-1} x < 1$

$\therefore$ $0 < e^x - 1 < 1$ from (B)

$1 < e^x < 2$ $\therefore$ a value of x exists.

hence a root in $(1, 2)$ interval

$(0 < x < \ln 2)$.

Consider $\frac{2}{\pi} < k < 1$

$-1 < k \tan^{-1} x < \frac{\pi}{2}$

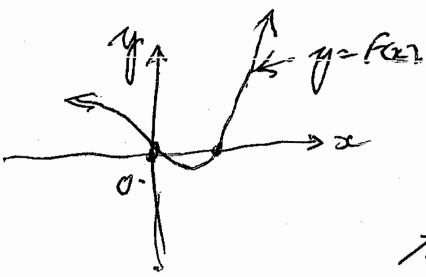
$0 < e^x < \frac{\pi}{2} + 1$ from (B)

$-\infty < x < \ln\left(\frac{\pi}{2} + 1\right)$

This interval contains the root
$x = 0$.

∴ There are two ROOTS. One at $x=0$ and ~~sub~~ the other to the right of the origin between 0 and ln 2. (depends on k)

∴ One st. point and two roots.



COMMENT

A very difficult question to get exactly correct.

Many used diagrams, without fully justifying their answer.

The key was (B) which made the question much easier.